DUSO Mathematics League 2014 - 2015

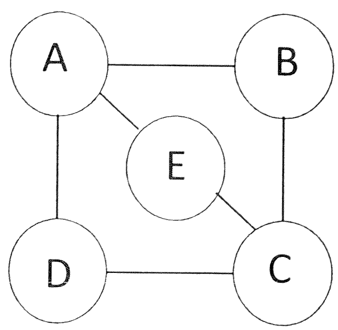
Contest #4. **Calculators are not permitted on this contest.**

Part I. **ALGEBRA I** *Time Limit: 10 minutes*

The word "compute" calls for an exact answer in simplest form.

4 - 1. The graph of $y = a^x + b$ passes through $(2, 5)$ and $(5, 239)$. Compute $a + b$.

4 - 2. In the accompanying picture, each of the five circles is colored red, yellow, or blue. No two circles which are connected by a line segment are the same color. Compute the number of distinct possible colorings.



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Part II. **GEOMETRY** *Time Limit: 10 minutes*

The word "compute" calls for an exact answer in simplest form.

4 - 3. In circle O, diameter $\overline{EC}$ is extended beyond E to A. Secant $\overline{ADB}$ is drawn with D and B on the circle. If $AE = 4$ and $AD = DB = 8$, compute OC.

4 - 4. The perimeter of parallelogram $ABCD$ is 30, and the altitudes to $\overline{AD}$ and $\overline{AB}$ have lengths 4 and 6 respectively. Compute the length of the longer side of $ABCD$.

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Part III. **ALGEBRA II / ADVANCED TOPICS** *Time Limit: 10 minutes*

The word "compute" calls for an exact answer in simplest form.

4 - 5. The graph of $y = A \sin Bx + D$ has a maximum at $(6, 8)$, a minimum at $(18, 2)$, and no maxima or minima between the points $(6, 8)$ and $(18, 2)$. Find the equation in the form $y = A \sin Bx + D$ where A, B, and D are in simplest form.

4 - 6. The graph of $y = ax^2 + bx + c$ passes through $(1, 4)$, $(2, -2)$, and $(4, -2)$. Compute the coordinates of the vertex of the parabola.

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Contest #4. **TEAM ROUND** **Calculators are not permitted on this contest.**

T-1. Compute all two-digit numbers such that the number is equal to twice the sum of its digits.

T-2. There are two base-10 numbers for which the base-9 representation is ABC and the base-11 representation is CBA. Compute the sum of these two base-10 numbers in base 10.

T-3. Compute the values of x that solve the following equation: $(x+5)^3 + (2x+4)^3 = (3x+9)^3$

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CONTEST #4.

SOLUTIONS

4 - 1. $\boxed{-1}$ Substituting yields $a^2 + b = 5$ and $a^5 + b = 239$. Subtracting, we have $a^5 - a^2 = 234$, which factors as $a^2(a^3 - 1) = 9 \cdot 26$, so $a^2 = 9$ and $a^3 - 1 = 26$. The value of a is 3 and the value of b is -4. Thus, $a + b = 3 + (-4) = \mathbf{-1}$.

4 - 2. $\boxed{48}$ Consider square A. There are 3 color choices for A, and therefore there are 2 color choices for E. As for B and D, they are either the same color or different colors. If B and D are different colors, then there are 2 choices for B and 1 for D, and all that must be done now is to ascertain the number of choices for C. If B, D, and E are all different colors, there is no possible choice for either A or C. If, however, two of the three are the same, then there is 1 choice for C. Thus, if B and D are different colors, there are $3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 12$ colorings. Now, consider the possibility that B and D are the same color. In this case, there are 2 choices for B and 1 for D. If B, D, and E are the same, then there are 2 choices for C and therefore $3 \cdot 2 \cdot 2 \cdot 1 \cdot 2 = 24$ colorings. If, on the other hand, B and E are different colors, then there is only 1 choice for C and therefore $3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 12$ colorings. The number of distinct possible colorings is $12 + 24 + 12 = \mathbf{48}$.

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4 - 3. $\boxed{14}$ Use the Power of a Point Theorem to find $OC = r$. $AE \cdot AC = AD \cdot AB$, so $4 \cdot (2r + 4) = 8 \cdot 16$ implies $r = \dfrac{32 - 4}{2} = \mathbf{14}$.

4 - 4. $\boxed{9}$ The longer side is $\overline{AD}$, since the shorter altitude is drawn to it; let $AD = x$ and $AB = 15 - x$. Then, finding the area of the parallelogram in two different ways, equate $4x = 6(15 - x)$. Solve to obtain $4x = 90 - 6x \to \mathbf{x = 9}$.

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4 - 5. $\boxed{\mathbf{y = 3\sin\frac{\pi}{12}x + 5}}$ The value of D is halfway between 8 and 2, so $D = \dfrac{8+2}{2} = 5$. The value of A is the vertical distance between the maximum and the midline, so $A = 8 - 5 = 3$. The period of the graph is $2 \cdot (18 - 6) = 24$, so $B = \dfrac{2\pi}{24} = \dfrac{\pi}{12}$. The equation of the graph is $y = 3\sin\dfrac{\pi}{12}x + 5$.

4 - 6. $\boxed{(3, -4)}$ Substituting $x-$ and $y-$coordinates into the general equation gives us three equations: $a + b + c = 4$, $4a + 2b + c = -2$, and $16a + 4b + c = -2$. Subtracting the first two equations yields $3a + b = -6$. Subtracting the second two equations yields $12a + 2b = 0 \to 6a + b = 0$. Subtracting these yields $3a = 6 \to a = 2 \to b = -12 \to c = 14$. The equation is $y = 2x^2 - 12x + 14$, whose vertex is at $\mathbf{(3, -4)}$.

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T-1. Compute all two-digit numbers such that the number is equal to twice the sum of its digits.
T-1Sol. $\boxed{18}$ Let the number be AB. Then, $10A + B = 2(A + B) \to 8A = B$. The only number that satisfies this is **18**.

T-2. There are two base-10 numbers for which the base-9 representation is ABC and the base-11 representation is CBA. Compute the sum of these two base-10 numbers in base 10.
T-2Sol. $\boxed{\mathbf{735}}$ The first sentence of the problem implies $81A + 9B + C = 121C + 11B + A$, which in turn implies $80A - 120C = 2B \to 20(2A - 3C) = B$. Since B is a multiple of 20 but $B < 9$, $B = 0$. Therefore, $2A - 3C = 0$. Since both A and C are less than 9, the only solutions are $A = 3$ and $C = 2$ or $A = 6$ and $C = 4$. The two base-9 numbers are 302 and 604, which convert to $3 \cdot 81 + 2 = 245$ and $6 \cdot 81 + 4 = 490$. Their sum is **735**.

T-3. Compute the values of x that solve the following equation: $(x + 5)^3 + (2x + 4)^3 = (3x + 9)^3$
T-3Sol. $\boxed{\{-\mathbf{5}, -\mathbf{3}, -\mathbf{2}\}}$ This equation is of the form $A^3 + B^3 = (A + B)^3$, which has solutions only if $A = 0$ or $B = 0$ or $A + B = 0$. Therefore, instead of expanding the brackets and proceeding to solve a difficult cubic, instead solve three linear equations to find $x + 5 = 0 \to x = -5$, $2x + 4 = 0 \to x = -2$, and $3x + 9 = 0 \to x = -3$. The solutions are $\{-\mathbf{5}, -\mathbf{3}, -\mathbf{2}\}$.

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